Review on Top. Spaces

Let X be a top. space which maybe has some nice properties. Let $\pi: Y \to X$ be a cont. map. this defines a sheaf on X via $\underbrace{U \mapsto \Gamma(U)}_{\text{sheaf of sections}} = \{ f: U \to Y \text{ cont.}, \pi \circ f = \mathrm{id} \}$. This is in a sense the "main example".

Exercise 14: (See Ex. II.1.14 in Hartshorne) Prove that for any sheaf of sets F on X, there is a top. space $[F]$ and a continuous map $\pi: [F] \to X$ with the following properties:

1) F is the sheaf of sections of π.
2) π is a local homeomorphism.
3) for all $x \in X$, $\pi^{-1}(x) = F_x$.

Usually, $[F]$ is not Hausdorff. For example if $x \in X$ and $F = \mathbb{Z}_x$ is a skyscraper sheaf at x, then $[F]$ is X itself, except over x, there are countably many points which cannot be separated.

Set $Sh(X)$ = abelian category of sheaves of abelian groups on X (there are some variants of this, depending on what X is). Note $Sh(X)$ has enough injectives (is a Grothendieck category), so we can take the right derived functors of $\Gamma(X, -)$ for an injective resolution:

$$0 \to F \to I^0 \to I^1 \to \cdots$$

$$\Rightarrow 0 \to \Gamma(X, F) \to \underbrace{\Gamma(X, I^0) \to \cdots}$$

$$\Rightarrow H^i(X, F) = H^i(\text{——}/\!/\text{——})$$

$H^\bullet$ is a cohomological functor, i.e. satisfies the axioms for a cohomology theory. If $f: X \to Y$ is a continuous map, we have the direct image functor: $f_* F(U) = F(f^{-1}(U))$, which maps $Sh(X) \to Sh(Y)$. Note if $Y = pt.$, $f_* = \Gamma(X, -)$, so this is a generalization. This is also left exact.

Taking right derived functors, $R^i f_* F = \mathcal{H}^i(f_* I^\bullet)$, where $\overset{\{f_* I^0 \to f_* I^1 \to \cdots\}.}{\overset{\shortparallel}{I^\bullet}}$ is an injective resolution of F, and are called higher direct images.

Def: Let $G: \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. An object A in $\mathcal{A}$ is G-acyclic if $R^i G(A) = 0$ for $i > 0$ (Assuming enough injectives).

Prop$^{(*)}$: To derive G in the above context, one can use resolutions of acyclic objects.

An example is a cont. map $f: X \to Y$, the flabby sheaves on X are acyclic with respect to f_*. Indeed an injective sheaf is flabby. The proof uses the extension functor, which we review later.

Can we visualize higher direct images? By def'n, $R^i f_* F$ is a sheafification of the presheaf $V \mapsto H^i(f^{-1}(V), F)$ on Y. One might notice this is a "cohomology of fibers".

Thm (Proper Base Change): If f is proper, $(R^i f_* F)_y \cong H^i(X_y, F|_{X_y})$.

Grothendieck Spectral Sequence

Let $\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$ be a left exact functor. Suppose $\mathcal{A}, \mathcal{B}$ have enough injectives. Suppose α takes injectives to β-acyclic. Then there is a spectral sequence, with

$$E_2^{p,q}(A) = R^q\beta(R^p\alpha(A)) \Rightarrow R^{p+q}(\beta\circ\alpha)(A).$$

<u>Example:</u> Leray Spectral Sequence: Consider $X \xrightarrow{f} Y \xrightarrow{p} pt.$ Let $F \in Sh(X)$, then the derived functors of $(p\circ f)_*$ is just cohomology. But the composition is $H^q(Y, R^pf_*F)$, and the spectral sequence shows they are the same.

Diagrammatically:

$$\begin{tikzcd}
\vdots & \vdots & & \\
H^0(Y, R^1f_*F) \arrow[rrd, "d"] & H^1(Y, R^1f_*F) \arrow[rrd, dashed, "d"] & & \\
H^0(Y, R^0f_*F) & H^1(Y, R^0f_*F) & H^2(Y, R^0f_*F) & \cdots
\end{tikzcd}$$

<u>Example:</u> Serre Spectral Sequence for Hopf fibration:

$$\begin{tikzcd}
f: S^{2n+1} \arrow[r, "\text{Hopf fibration}"] & \mathbb{CP}^n \\
\mathbb{C}^{n+1}\setminus\{0\} \arrow[u, phantom, "\cap"] \arrow[ur, "\mathbb{C}^*"'] &
\end{tikzcd}$$

Let $F = \mathbb{Z}_{S^{2n+1}}$. Then $E_2^{p,q} = H^q(\mathbb{CP}^n, R^pf_*F)$. Since f is proper, and fibers are circles. Hence:

$$(R^pf_*F)_y = H^p(S^1_y, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & p = 0,1 \\ 0, & \text{else.} \end{cases}$$

Since f is a locally trivial fibration, all other higher direct images are locally constant. But $\pi_1(\mathbb{CP}^n) = 0 \Rightarrow$ They are actually constant. Thus:

$$R^pf_*F = \begin{cases} F, & p = 0,1 \\ 0, & \text{else.} \end{cases}$$

Putting this together, the $E_2^{p,q}$ page is

$$\begin{tikzcd}
H^0(\mathbb{CP}^n, F) \arrow[rrd, "d_2"] & H^1(\cancel{\mathbb{CP}^n, F}) & H^2(\mathbb{CP}^n, F) \arrow[rd, "d_2"] & --- \arrow[rd, "d_2"] & H^{2n}(\mathbb{CP}^n, F) \\
H^0(\mathbb{CP}^n, F) & H^1(\cancel{\mathbb{CP}^n}, F) & \underset{\substack{\rotatebox{90}{$\in$} \\ c_1(\mathcal{O}(-1))}}{H^2(\mathbb{CP}^n, F)} & \cdot\, - & H^{2n}(\mathbb{CP}^n, F)
\end{tikzcd}$$

computing gives the result.

<u>Exercise 15:</u> Use the Serre spectral sequence to compute the cohomology of ΩS^3.

<u>Exercise 16:</u> Prove proposition $(*)$ above. (Hint: Spectral sequence).

Given $f: X \to Y$, get exact functor $f^*: Sh(Y) \to Sh(X)$. An (f_*, f^*) form an adjoint pair. In addition, $f_!: Sh(X) \to Sh(Y)$, with $W \subset Y$ open:

$$(f_!F)(W) = \{ s \in f_*F(W) \mid f: \text{supp}(s) \to Y \text{ is proper} \}.$$

If $j: U \hookrightarrow X$ is an open embedding, $j_!$ is extension by zero.

<u>Prop:</u>

1) There is a map $f_! \to f_*$.
2) If f is proper, $f_! = f_*$.
3) $f_!$ is left exact, with derived functors higher direct image w/ compact support (if $Y = pt$, this is cohomology with compact support.)

Derived Categories

Let $\mathcal{A}$ be an abelian category. Define $K(\mathcal{A})$ by objects being complexes in $\mathcal{A}$, and morphisms being chain maps, if localized at quasi-isomorphisms, this gives the Derived category $D(\mathcal{A})$.

Main Property: $D(\mathcal{A})$ is triangulated.

If $f: \mathcal{A} \to \mathcal{B}$ is left exact between abelian categories, we get right derived functors $R^i f: \mathcal{A} \to \mathcal{B}$. However we can define $Rf: D(\mathcal{A}) \to D(\mathcal{B})$ by sending $0 \to X \to 0 \to 0$ to the cohomology of the complex $f(0 \to X \to 0 \to 0 \dots)$.

If we have $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$, we can get derived functors $D(\mathcal{A}) \xrightarrow{Df} D(\mathcal{B}) \xrightarrow{Dg} D(\mathcal{C})$, and one can see $D(g \circ f) = Dg \circ Df$ (a bit nicer then the Grothendieck spectral sequence!).

Setting $D(X) = D(sh(X))$, X a top. space, and $f: X \to Y$ continuous, we get:

$$\left.\begin{array}{l} Df_* : D(X) \longrightarrow D(Y) \\ f^* : D(Y) \longrightarrow D(X) \end{array}\right\} \text{Still an adjoint pair}$$

$\uparrow$ already exact

Also $f_! : Sh(X) \to sh(Y)$ produces $Df_! : D(X) \to D(Y)$. The adjoint functor is $f^!$ and is <u>only</u> a functor of derived categories. $(Rf_!, f^!)$ is the adjoint pair. We have two main distinguished triangles:

$$\begin{tikzcd} \overset{\text{closed}}{Z} \arrow[r, hook, "i"] & X & \overset{\text{open}}{U} \arrow[l, hook', "j"'] \end{tikzcd}$$

$\forall F \in Sh(X)$, $Ri_! \circ i^! F \longrightarrow F \longrightarrow Rj_* \circ j^* F$, and $Rj_! \circ j^! F \longrightarrow F \to Ri_* \circ i^* F$. So we get long exact sequences of hypercohomology:

$$\dots \to \mathbb{H}^0(X, Ri_! \circ i^! F) \longrightarrow \mathbb{H}^0(X, F) \longrightarrow \mathbb{H}^0(X, Rj_* \circ j^* F) \longrightarrow \dots \quad \text{(same for the other)}$$

Example:
Fix $F = \mathbb{Q}_X$. Then the second Δ gives $0 \longrightarrow j_! \mathbb{Q}_U \to \mathbb{Q}_X \longrightarrow i_* \mathbb{Q}_Z \to 0$, which is just a sequence of sheaves. So if we wanted $H^i(X, \mathbb{Q}_X)$, we can compare this to $H^i(U, \mathbb{Q}_U)$ and $H^i(Z, \mathbb{Q}_Z)$. (Actually the first Δ is better for this.)

Čech Cohomology

See "Differential forms in Algebraic Topology".

Thm: If X is paracompact, then $\check{H}^i(X, F) \cong H^i(X, F)$.

Exercise: Prove Ex. III. 4.11 in Hartshorne.

Example: Let X be a scheme and $F \in QCoh(X)$. Take X separated and $U_i = \operatorname{Spec} A_i$ an open covering of affines. Then $H^i(U_i, F) = 0$ for $i > 0$, and see the above exercise.

Constructible Sheaves

Assume a sheaf is a sheaf of vector spaces (over $\mathbb{Q}$).

Exercise 18: Let X be a "nice" connected topological space (say a manifold). Prove that there is an equivalence of categories:

$$\left\{ \begin{matrix} \text{locally constant sheaves} \\ \text{on } X \end{matrix} \right\} \longleftrightarrow \left\{ \pi_1(X)\text{-modules} \right\}.$$

Def: A stratification of X is a decomposition into finitely many pieces $X = \amalg X_i$, such that

a) each X_i is a topological manifold,

b) each X_i is locally closed (open in $\overline{X_i}$),

c) $\forall i$, $\overline{X_i}$ is a union of other strata.

d) A condition on the topology (look this up).

If X is a complex alg. variety, a stratification is: $X^1 = X_{\text{smooth}}$, $X^2 = (X \setminus X_1)_{\text{smooth}}$, etc. This can be refined to a Whitney stratification.

Def: Let X be stratified and $F \in Sh(X)$ is constructible if for all i, $F|_{X_i}$ is locally constant of finite rank.

We take (typically), $X = \mathbb{C}$-alg. variety, X_i = locally closed $\mathbb{C}$-alg. variety. We attach to this the category:

Def: Let X be a $\mathbb{C}$-alg. variety. A sheaf F on X^{an} is constructible if it is constructible with respect to some stratification of X.

Let $D^b_c(X) \subset D(X)$ be the full subcategory of complexes of sheaves with bounded cohomology ($\mathcal{H}^i(C^\bullet) = 0$ for $|i| >> 0$) and each $\mathcal{H}^i(C^\bullet)$ is constructible.

Thm: Let $f: X \to Y$ be a morphism of $\mathbb{C}$-alg. varieties. Then the functors Rf_*, f^*, $Rf_!$, $f^!$ preserve the constructible categories. They also behave well with base change (look this up).

Thm: (Verdier Duality) There is an object $\mathcal{D}_X \in D(X)$ called the dualizing complex, such that the contravariant functor $D_X = R\mathcal{H}om(-, \mathcal{D}_X) : D(X) \longrightarrow D(X)$:

1) Preserves the constructible category

2) $D_X^2 = Id$ on $D^b_c(X)$.

3) If X is smooth, $D_X = \mathbb{Q}_X[2 \cdot \dim X]$.

Thm: There are isomorphisms of functors for $f: X \to Y$:

$$\begin{tikzcd} D^b_c(X)^{op} \arrow[r, shift right, "Rf_!"'] \arrow[d, "D_X"] & D^b_c(Y)^{op} \arrow[l, shift right, "f^!"'] \arrow[d, "D_Y"] \\ D^b_c(X) \arrow[r, shift left, "Rf_*"] & D^b_c(Y) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

So $f^! \longleftrightarrow Rf_*$

$Rf_! \longleftrightarrow f^*$.

Example: Take a smooth $\mathbb{C}$-alg. variety X, and set $\dim_{\mathbb{C}} X = n$. Then $\mathcal{D}_X = \mathbb{Q}_X[2n]$. Take $F = \mathbb{Q}_X$, and $p: X \to pt$. Then applying the above:

$$D_{pt} \circ Rp_!(\mathbb{Q}_X) = Rp_* \circ D_X(\mathbb{Q}_X) = Rp_* \circ \mathcal{D}_X = Rp_* \mathbb{Q}_X[2n].$$

The right hand side is (i^{th} cohomology) $H^{i+2n}(X, \mathbb{Q})$, and the left hand side is $H_c^{-i}(X, \mathbb{Q})^\vee$, which is the classical Poincare Duality! ($H^{2n-j}(X, \mathbb{Q}) \cong H^j_c(X, \mathbb{Q})^\vee$).

Note we only used constancy at $D_X(\mathbb{Q}_X) = \mathbb{Q}_X[2n]$. So this gives a hint at how to formulate this more generally (if say, X is not smooth), just work with objects s.t. $D_X F = F[2n]$.

<u>Thm: Poincare Duality for Singular Varieties</u>

Let X be a $\mathbb{C}$-alg. variety. There is a canonical object $F \in D^b_c(X)$ s.t.

1) $D_X F = F[2 \cdot \dim X]$
2) $F|_{X^{\text{sm}}} = \mathbb{Q}_X$
3) $F = IC_X$ = intersection cohomology complex on X.

See Asterisque volume 100 for a discussion.

Étale Sheaves

Let X be a scheme (with our usual conventions).

<u>Def</u>: Let X_{et}, called the (small) étale site, be the category consisting of schemes over X with structure map étale, and morphisms X-morphisms.

A covering of U is a collection $\{U_i \xrightarrow{g_i} U\}$ of étale maps with $\bigcup g_i(U_i) = U$. Similarly the Zariski / Flat site are objects $Y \to X$ open embeddings / flat + LoFT, and similar coverings (faithfully flat + q-compact in the flat case).

Have "morphisms" of sites (colloquially, not functors) $X_{fl} \to X_{et} \to X_{zar}$.

A presheaf on X_{et} is just a contravariant functor $P: X_{et} \to \mathcal{C}$. The sheaf condition says for all coverings $\{U_i \to U\}$, the diagram

$$P(U) \longrightarrow \prod P(U_i) \rightrightarrows \prod P(U_i \times_U U_j)$$

is an equalizer diagram.

<u>Examples</u>:

0) G be a discrete grp. Then $G_X = X \times G = \coprod_{g \in G} X_g$ is a group scheme. Gives a sheaf $F(Y) = \mathrm{Hom}(Y, G_X)$ (sometimes called G_X). Get similar examples for additive/ multiplicative grps.

1) Let $F \in \mathrm{QCoh}(X)$. Define a presheaf $W(F): \mathrm{Sch}/X \to \mathrm{Ab}$, by

$$W(F)(Y \xrightarrow{a} X) = \Gamma(Y, a^* F).$$

<u>Claim</u>: $W(F)$ is a sheaf on the étale site (flat site).

<u>Exercise 14</u>: Prop. 1.5 in EC, chap 2.

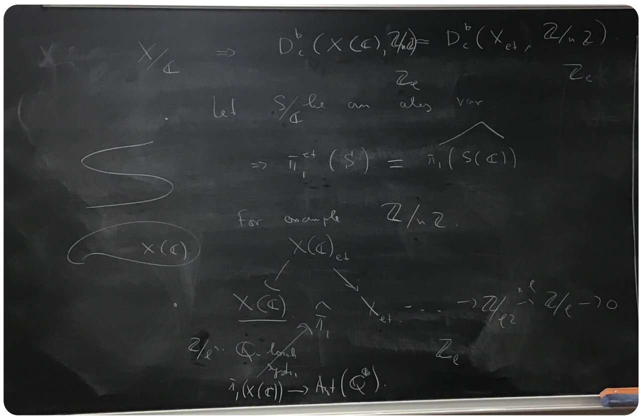
The exercise proves $W(F)$ is a sheaf.

So given a sheaf F (étale sheaf) on X_{et}, we want a notion of a stalk. We make the natural definition: given a geometric point $\bar{x} \in X$;

$$F_{\bar{x}} = \lim_{\substack{\bar{x} \to U \\ \searrow X \swarrow \text{et.}}} F(U)$$

Which can be related to the henselization.

Now taking $\bar{X} = X \times_k \bar{k}$, we get that $G = Gal(\bar{k}/k)$ acts on $\bar{X}$, and hence on $\bar{X}_{et}$ by autoequivalences. Given a sheaf $F \in S(\bar{X}_{et})$, suppose $\sigma^* F \cong F$ for all $\sigma \in G$. Then we get a morphism $\theta : H^i(\bar{X}_{et}, \sigma^* F) \longrightarrow H^i(\bar{X}_{et}, F)$ which is an automorphism. This can give (via several difficult conjectures) to Hodge Theory!



Def: Let G be a finite group. A finite étale morphism $f: Y \to X$ is a Galois covering if

a) G acts on Y/X,

b) If $G_X = G \times X$, then we have a canonical map $Y \times G_X \longrightarrow Y \times_X Y$, and this should be an isomorphism.

Exercise 20: Let $k \subset k'$ be a finite Galois extension with $G = Gal(k'/k)$. Put $f: Y \to X$, w/ $Y = \operatorname{Spec} k'$, $X = \operatorname{Spec} k$. Prove f is a Galois covering.

Prop: Let $f: Y \to X$ be a Galois covering with group G. Let F be a presheaf on X_{et}. The group G acts on $F(Y)$, with diagram

$$\begin{tikzcd}(*).\ F(X) \arrow[r, "f^*"] & F(Y) \arrow[r, shift left, "(1,1,\dots,1)"] \arrow[r, shift right, "(\sigma_1,\dots,\sigma_n)"'] & F(Y)^n\end{tikzcd}, \quad G = \{\sigma_1, \dots, \sigma_n\}.$$

Then if F is a sheaf, this diagram is exact (equaliser).

Proof 1.4, chap 2 in EC. ▨

As an application, lets look at étale sheaves on $\operatorname{Spec} k$, $S((\operatorname{Spec} k)_{et})$. If G is a profinite group ($G = \varprojlim G/G_i$, G/G_i finite). A G-module M is discrete if for all $m \in M$, G_m is a discrete group. Thus if $G = Gal(\bar{k}/k)$, we have an abelian category of discrete G-modules. Claim: $S((\operatorname{Spec} k)_{et}) \simeq$ this category.

Let $F \in S((\operatorname{Spec} k)_{et})$. We have a geometric pt. $\bar{x} = \operatorname{Spec} \bar{k} \longrightarrow \operatorname{Spec} k = X$. Taking the stalk, we only need to look @ the system of k'/k finite Galois. Checking the def'n:

$$F_X = \bigcup_{\substack{k'/k \\ \text{fin. Gal.}}} F(\operatorname{Spec} k') \circlearrowleft Gal(\bar{k}/k)$$

and is a discrete module!

Conversely, if M is a discrete Galois module, let $U \to \operatorname{Spec} k$ be étale $\Rightarrow U = \coprod_{\substack{k'/k \\ \text{fin. sep.}}} \operatorname{Spec} k'$

Take $F(U) = \bigoplus_{k/\kappa} M^{Gal(\bar{k}/\kappa)}$. Certainly a presheaf, and an application of the proposition can show its a sheaf.

Presheaves and Sheaves

Theorem: The inclusion $f : S(X_{et}) \longrightarrow P(X_{et})$ is left exact, and has a left adjoint exact functor, sheafification: $a : P(X_{et}) \longrightarrow S(X_{et})$. We see

a) P and aP have the same stalks.

b) TFAE:

- i) $0 \to F \to F' \to F''$ is exact in $S(X_{et})$,
- ii) $\forall\, U/X \in X_{et}$, $0 \to F(U) \to F'(U) \to F''(U)$ is exact in grps,
- iii) $\forall\, \bar{x} \to X$, $0 \to F_{\bar{x}} \to F'_{\bar{x}} \to F''_{\bar{x}}$ is exact.

c) TFAE:

- i) $\phi : F \to F'$ is a surjection in $S(X_{et})$,
- ii) $\forall\, U/X \in X_{et}$, $\forall s \in F(U)$, there is a covering $\{U_i \to U\}$ + elements $s_i \in F(U_i)$ such that $\phi_{U_i}(s_i) = res_{U,U_i}(s)$.
- iii) $\forall\, \bar{x} \to X$, $F_{\bar{x}} \to F'_{\bar{x}}$ is surj.

Examples:

a) M an abelian grp. P_M be the constant presheaf $P_M(U/X) = M$. Define $F_M = aP_M$, as the constant sheaf.

b) Recall sheaf $\mathbb{G}_m$ on X_{et}, represented by $\operatorname{Spec} \mathbb{Z}[t, t^{-1}] \times_{\operatorname{Spec}\mathbb{Z}} X$. There is an endomorphism $t \mapsto t^n$, denoted $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$. Lets look at the kernel.

$$\begin{tikzcd}
\mathbb{G}_m(U) \arrow[r, phantom, "="] \arrow[d, "n"'] & \Gamma(U, \mathcal{O}_U)^* \arrow[d, "(\;)^n"] \\
\mathbb{G}_m(U) \arrow[r, phantom, "="] & \Gamma(U, \mathcal{O}_U)^*
\end{tikzcd}$$

Kernel is functions whose n^{th} power is one. Denote this by $\mu_n(U)$, which is another sheaf, and is represented by $\mu_n = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]/(t^n - 1)$.

So we have an exact sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$. Is it surjective?

Claim: If $(n, \operatorname{char} X) = 1$ $(\Leftrightarrow \forall x \in X, (n, \operatorname{char} k(x)) = 1)$, then its surjective.

Indeed let $U = \operatorname{Spec} A$, $a \in A^*$. Need $V \to U$ étale s.t. $\exists\, b \in \Gamma(V, \mathcal{O}_V)^*$, $b^n = a$. Take $V = \operatorname{Spec} B$, $B = A[t]/t^n - a$. But if we have the conditions on the characteristic, this is standard étale!

Note that in the flat topology, this sequence is always exact!

Def: The short exact sequence $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$ is called the Kummer sequence.

One should think of this as the étale analog of the exponential sequence.

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X_{et}, \mathbb{G}_m) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^2(X_{et}, \mu_n) \arrow[r] & \cdots \\
& H^1(X_{Zar}, \mathbb{G}_m) \arrow[d, phantom, "\cong" rotate=90] & & \\
& \operatorname{Pic} X \; ! & &
\end{tikzcd}$$

Will prove later

analog of the 1st chern class.

Suppose X/k, $\bar{k} = k$. Then $\mu_n \cong (\mathbb{Z}/n\mathbb{Z})_X$ non-canonically. Indeed choose an n^{th} root of 1 in k, ξ. Then $\forall\, U/X$, $\mu_n(U) = \{\xi, \xi^2, \dots, \xi^n\} = \mathbb{Z}/n\mathbb{Z}$.

Classical Kummer Theory says:

Thm: Let k be a field containing n^{th} roots of 1. Let L/k be a cyclic Galois extension with Galois grp $|G| = n$. Then $L = k(\alpha)$, with $\alpha^n = a \in k$.

Related to Hilberts Theorem 90. There is an additive analogue as well. Then the analogue of the exponential sequence in $X/\operatorname{Spec}\mathbb{F}_p$ is:

$$0 \longrightarrow \operatorname{Spec}\left(\mathbb{F}_p[t]/t^p - t\right) \longrightarrow \mathbb{G}_a \xrightarrow{F - id} \mathbb{G}_a \longrightarrow 0,$$

where F is the frobenius, $\mathbb{G}_a$ the additive group scheme.

Exercise 21: Prove that the $\operatorname{Spec}\mathbb{F}_p$ - group scheme $\operatorname{Spec}\left(\mathbb{F}_p[t]/t^p - t\right)$ is isomorphic to the group scheme $(\mathbb{Z}/p\mathbb{Z})_{\operatorname{Spec}\mathbb{F}_p}$.

Direct & Inverse Images of Étale sheaves

Let $\pi : X' \longrightarrow X$ be a morphism of schemes and $F \in S(X'_{et})$. Define a presheaf $\pi_* F$ on X_{et} by declaring: $\pi_* F(U) = F(X' \times_X U)$, and will be an actual sheaf (note it is left exact).

The direct image functor has a left adjoint $\pi^* : S(X_{et}) \longrightarrow S(X'_{et})$. This will be
$\pi^* F = \varinjlim_{\substack{U' \to U \\ \searrow X'^{/et}}} F(U)$.

One can also check (at geometric points): $(\pi^* F)_{\bar{x}'} = F_{\pi(\bar{x}')}$ and $(\pi_* F)_{\bar{x}} = \Gamma(X' \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}, j^* F)$ where $j : X' \times_X \operatorname{Spec} \mathcal{O}^h_{X,x} \longrightarrow X'$.

Examples:

1) Take $X = \operatorname{Spec} k$, $X' = \operatorname{Spec} k'$, with k'/k separable. (finite, Galois.) We know $S((\operatorname{Spec} k)_{et})$ are discrete Galois-modules. Then the inverse image functor takes the restricted representation.

Exercise 22: Prove the first part of 3.7 in EC (chap 2). Uses Exercise #9.

Now we have a short exact sequence of étale sheaves:

$$0 \to \mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,X} \to i_{*X}\, \mathbb{Z} \to 0,$$

where g includes the generic pt.